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Pure Spinors, Free Differential Algebras, and the Supermembrane[†]

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Abstract

The lagrangian formalism for the supermembrane in any 11d supergravity background is constructed in the pure spinor framework. Our gauge-fixed action is manifestly BRST, supersymmetric, and 3d Lorentz invariant. The relation between the Free Differential Algebras (FDA) underlying 11d supergravity and the BRST symmetry of the membrane action is exploited. The "gauge-fixing" has a natural interpretation as the variation of the Chevalley cohomology class needed for the extension of 11d super-Poincaré superalgebra to M-theory FDA. We study the solution of the pure spinor constraints in full detail.

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1 Introduction

The strong regime of string theory is usually denoted M-theory. However, up to now, the underlying fundamental theory and the degrees of freedom are still unknown. Some indications coming from the low-energy effective action, accurately described by 11-dimensional supergravity, from the presence of extended objects in string theory such as the supermembrane and the M5-brane (from which all D-branes can be obtained by dimensional reduction), and from the superalgebra in 11 dimensions, pointed out that a plausible candidate for the fundamental theory is the theory of the *supermembrane*. This theory has been discovered in [1] and, since then, several studies have been performed to understand if it really has all the necessary features to describe M-theory. We do not present here a review and we refer to [3] for a complete account on the subject. Anyway, we need to remind the reader of some basic facts about the supermembrane.

The supermembrane is a theory of maps from a (2+1)-worldvolume to a (10+1)-dimensional target space. When the membrane moves in a flat superspace, the fundamental mathematical quantities are the supersymmetric line elements $\Pi_i^{\underline{a}} = e_i^\mu(\xi) (\partial_\mu X^{\underline{a}} + i \bar{\theta} \Gamma^{\underline{a}} \partial_\mu \theta)$ and $\psi_i^\alpha = e_i^\mu(\xi) (\partial_\mu \theta^\alpha)$ where $i = 1, 2, 3$ are the worldvolume flat indices, $(e_i^\mu(\xi))$ is the inverse dreibein) $\underline{a} = 0, \dots, 10$ are the flat target space indices and $\alpha = 1, \dots, 32$ are the indices for the spinorial representation of $SO(1, 10)$. The coordinates $X^{\underline{a}}$ and θ^α define a local basis in the superspace of the target space. The theory written in terms of the basic supersymmetric quantities is manifestly Lorentz and supersymmetric invariant. In addition, it is invariant under local diffeomorphism on the worldvolume and under an infinite reducible gauge symmetry known as κ -symmetry (see [1, 2] and references therein for details). These gauge symmetries remove the correct number of bosonic and fermionic degrees of freedom to have a manifestly supersymmetric spectrum [8] and they are crucial in the quantization procedure.

The symmetries (rigid and local), the action and the supersymmetry of the supermembrane theory are very similar to those of the superstring in the Green-Schwarz (GS) formalism and therefore we can use it as an example. The GS superstring is characterized by a set of fermionic constraints $d_{\underline{\alpha}}$ of the first and second class type. These constraints are entangled together in a single quantity

$$d_{\underline{\alpha}} \equiv p_{\underline{\alpha}} - \frac{\partial \mathcal{L}_{GS}}{\partial_0 \theta^{\underline{\alpha}}} = 0 \quad (1.1)$$

where \mathcal{L}_{GS} is the Green-Schwarz action and it is impossible to separate them locally without breaking Lorentz invariance and supersymmetry.

To follow the conventional quantization strategy, one has to use the BRST quantization technique for the first class constraints and to replace the Poisson brackets by the Dirac bracket to take into account the second class constraints. Instead, following Berkovits [4], one defines a BRST-like charge by

$$Q = \int_{\xi_0=t} d^{d-1} \xi (\lambda^{\underline{\alpha}} d_{\underline{\alpha}}) (\xi), \quad (1.2)$$

where $\lambda^{\underline{\alpha}}$ are commuting spinors and ξ^μ are the worldvolume coordinates. The integration is extended over the spatial coordinates. Here we make no distinction between first and second class constraints and the Poisson brackets are used to compute the commutation relations.

This implies that for a nilpotent BRST symmetry, some constraints on the ghost fields $\lambda^\alpha(\xi)$ are necessary and the latter are now known in the literature as *pure spinor constraints*.

During the last six years, the quantization of superstring according to [4] has been studied and several results have been already achieved. Still, the formalism is not yet complete and several issues need to be understood and clarified. One of these issues is the geometric structure underneath: even if the formalism seems to give consistent results, it is rather important to spell out its geometrical structure in particular to study $M2$ -branes in a generic 11d supergravity background. As an example, the geometrical formulation of superstrings and supermembranes using κ -symmetry like in ref. [1] permits useful expressions for any worldvolume and target space background.

The pure spinor formalism for the superstrings has been adapted to supermembranes in [5]. There it is shown that starting from the original action of [1], one can derive the fermionic constraints $d_A(\xi)$ from which the BRST charge Q can be constructed. The ghost fields $\lambda^A(\xi)$ carry an index in $\text{Spin}(32)$, they are commuting scalars and they satisfy to pure spinor conditions

$$\bar{\lambda}\Gamma^a\lambda = 0, \quad \bar{\lambda}\Gamma^{ab}\lambda\eta_{\underline{a}\underline{b}}\Pi_I^{\underline{b}} = 0, \quad \bar{\lambda}\partial_I\lambda = 0, \quad (1.3)$$

where the index I only runs over the spatial directions of the worldvolume.

In [5] the supermembrane is studied using the Hamiltonian approach [6] and therefore it is not manifestly covariant on the worldvolume. Nevertheless, it is shown that by freezing the transverse degrees of freedom of the membrane, the action reduces to a superparticle action whose spectrum is identified with 11d supergravity (and the equations of motion are given at the linearized level). Therefore, the pure spinor supermembrane action has all symmetries gauge-fixed and it provides the starting point for a complete analysis at the quantum level.

Unfortunately, even in this new framework the gauge fixed action is interacting and it cannot be further simplified. Therefore, besides the first massless level, the full spectrum of the membrane is still unknown (see [7] and [8] for a review on the spectrum of the supermembrane in the semiclassical regime and on the light-cone gauge). It is however useful to note that some computations can be indeed performed explicitly. By analyzing the low-energy spectrum, one can discover that the states are organized in such a way that some tree level and one-loop amplitudes can be constructed (almost algebraically by zero mode saturation rules [13]) and computed in the approximation of zero transverse fluctuations ([9] and [10]).

In [5] (following [11]), the pure spinor action is obtained by a BRST-like approach where the classical action is replaced by the gauge fixed action by adding a BRST-exact term

$$S_{\text{classical}} \rightarrow S_{\text{classical}} + \mathcal{S} \int d^d\xi \Phi_{\text{gauge}}(\xi) \quad (1.4)$$

where the *gauge fermion* $\Phi_{\text{gauge}}(\xi)$ is a local functional of the fields of the theory and we denote by \mathcal{S} the functional differential BRST operator. In the conventional BRST approach the quantum action is invariant under the BRST symmetry since the classical action is invariant under the gauge symmetry and the BRST operator is nilpotent. In the pure spinor string/membrane theory, each single term is not invariant, but only the sum is such. This is possible because of the non-invariance of the classical action and of the non-nilpotency of the BRST charge (this point will be elucidated in the text).

The form of the BRST charge is obtained by using the Hamiltonian formalism, and therefore the action of the charge on each field is determined by the Poisson brackets of the charge with the corresponding fields.

In the present work, we rather start from a Lagrangian approach. In that case the action of the BRST charge on the conjugate momenta is not fixed *a priori*, but it should be determined from the consistency of the formalism. Therefore the starting point is rather different and, as we are going to emphasize leads to a full fledged geometrical interpretation of the action, of the BRST transformations and also of the pure spinor constraints.

In order to introduce the reader to our viewpoint we have to recall some fundamental developments in the general geometrical understanding of supersymmetric field theories which are by now quite old, but they are quite essential to our present arguing. The first is the geometrical decoding of local supersymmetry itself.

1.1 Geometry of Supersymmetry

In every supergravity theory for any number of permitted space–time dimensions ($2 \leq D \leq 11$) and for any number of permitted supersymmetry charges \mathcal{N}_{SUSY} a supersymmetry transformation is nothing else but a *Lie derivative* $\mathcal{L}_{\vec{\epsilon}}$ along a tangent vector $\vec{\epsilon}$ of fermionic type. The crucial and fascinating point is however that the superfields describing the geometry of superspace on which such Lie derivatives act – namely the supervielbein and the super-connection appropriate to the considered case – are not free, rather they have to satisfy a unique set of constraints similar to Cauchy–Riemann conditions. These are named the *rheonomic conditions* [20] and encode all the symmetries together with the classical dynamics of any supergravity. In short, they are the very definition of the theory. Mathematically the rheonomic conditions (i.e. *rheonomic principle*) are expressed by the requirement that the components of all superspace curvature components along fermionic directions should be expressed as linear combinations of the curvature components along bosonic directions. Obviously this must be done in a way compatible with the fulfillment of Bianchi identities and it turns out that the solution to such a problem is always unique. It determines the explicit form of the supersymmetry transformations on all fields of the theory and at the same time it also determines the dynamics. Indeed, the rheonomic conditions imply some constraints on the bosonic curvature components that are interpreted as the field equations (Einstein equation, Rarita-Schwinger equations and so on). Therefore, supergravity theories are completely determined by the choice of the superalgebra plus the construction of the unique *rheonomic parametrization* of its curvatures. In the case of higher dimensional supergravity superalgebras are replaced by the larger category of Free Differential Algebras (See appendix A for a short review of the concept.)

The second fundamental advance relevant to our present discussion is the geometrical interpretation of κ -symmetry in p -brane theories. Indeed it was realized that this is no new exotic symmetry which has to be invented case by case rather it is nothing else, but the same supersymmetry which is already determined by the *unique rheonomic parametrization* of the supergravity curvatures describing the ambient superspace geometry in which the p -brane evolves. The only novelty is a restriction on the fermionic tangent vector $\vec{\epsilon}$ along which one can calculate the Lie derivative $L_{\vec{\epsilon}}$. This restriction is actually encoded in a projection operator \mathcal{P}_q , which applied to $\vec{\epsilon}$ enforces the operation $L_{\mathcal{P}_q \vec{\epsilon}}$ to be parallel to

the world volume evolution. This viewpoint on κ -symmetry is very fruitful. It originated from work done in [19] and was extended and fully merged with rheonomy in [21, 27]. In this latter paper, in particular, it was observed that the explicit form of κ -symmetry, namely the projector \mathcal{P}_q can be easily derived from a first-order action coupled to a generic supergravity background and the general rules were established how to write κ -symmetric p -brane actions in first order rheonomic formalism. The case of the M2-brane was spelled out explicitly [21] and it was shown to lead to the anti de Sitter supersingleton in case the background is chosen to be $\text{AdS}_4 \times \mathbb{S}^7$.

The third advance in geometrical understanding concerns the geometrical decoding of the BRST quantization of supergravity theories. The first step due to [34] consists of constructing a double elliptic complex in which the exterior derivative operator d in superspace and the BRST charge $Q = \mathcal{S}$ are merged into a new nilpotent operator $\mathbf{d} = d + \mathcal{S}$, while all the super p -forms are extended to super p ghost-forms, the extra components of which are the ghost-fields. In order to obtain explicit BRST-transformations, however, one has to implement suitable conditions on the curvatures of the ghost-forms, similar to the rheonomic conditions imposed on classical super curvatures in order to define supergravity. In [33] it was discovered that the right answer encompassing the correct BRST algebra for any supergravity theory is provided by a very simple and general principle which states the following *The rheonomic parametrization of the ghost-form supercurvatures is formally identical, mutatis mutandis, to the rheonomic parametrization of the classical supercurvatures*. In other words it suffices to keep the same form for all the curvature components and simply substitute the extended ghost-form where the corresponding classical form appeared.

1.2 Outlook

In the present paper, relying on the above results as starting point we take a leap forward and we show where the pure spinor constraints come from. They emerge from a constrained BRST algebra which is obtained from the ordinary BRST algebra of $D = 11$ supergravity by imposing that the diffeomorphism ghosts and the gauge ghosts of the three-form $\mathbf{A}^{[3]}$ should be zero, namely by unfreezing the target space diffeomorphisms and the three-form gauge transformations. The logical steps are the following ones:

1. The superPoincaré algebra in $D = 11$, uniquely defines its own extension to a Free Differential Algebra FDA_{11} via its cohomology (see appendix A).
2. The FDA_{11} via its unique rheonomic parametrization defines classical $D=11$ supergravity and, in conjunction with the classical supermembrane action, defines the κ -symmetry transformations of the latter.
3. The rheonomic parametrization of the FDA_{11} curvatures, uniquely extended to ghost-forms define the unconstrained BRST-algebra of $D = 11$ supergravity. Under these BRST-transformation the κ -symmetric classical supermembrane action is not invariant.
4. The constraint that the diffeomorphisms ghosts should be zero, inserted in the ordinary BRST algebra uniquely determines, from consistency, a set of constraints on

the superghosts that are identified as the *primary pure spinor constraints*. The supermembrane action can be made invariant against this constrained BRST algebra by adding to the classical action a new uniquely determined *gauge fixing part* of the form $\mathcal{S} \int d^d \xi \Phi_{gauge}(\xi)$ which is actually related to the very same cohomology class which originated the FDA extension and all the rest.

So, we start from the target space BRST-algebra and we set to zero both the translation and the gauge symmetry ghosts of the three-form and we require that the algebra still closes. Explicitly this implies that the supersymmetry ghost λ should satisfy the following constraints:

$$\bar{\lambda} \Gamma^a \lambda = 0, \quad \bar{\lambda} \Gamma^{ab} \lambda \eta_{\underline{a}a'} \eta_{\underline{b}b'} \Pi_{[i}^{a'} \Pi_{j]}^{b'} = 0. \quad (1.5)$$

It is easy to compare these constraints with those found using the Hamiltonian formalism: these are weaker, but the first constraint is just the usual 11d pure spinor constraint for the superparticle.

From supersymmetry, we can easily deduce the BRST transformation rules for the fields X^a, θ^a, \dots and we can compute the BRST variation of the action. This variation turns out to be proportional to the gravitino form Ψ and as we already stated it should be cancelled by a suitable gauge fixing term. The gauge fixing term is guessed of the form $\mathcal{S}(\text{something})$ and, therefore, its variation comes only from the nilpotency of the charge itself. We show that it is indeed possible to find a suitable gauge fixing and a BRST variation of the antighost field to have an invariant action.

As a second step, we have to check the nilpotency and the consistency condition for the BRST transformation rules. We found that under a stronger form of the constraints derived from the supersymmetry algebra the BRST algebra closes and we have a consistent framework. It is important to note that the constraint we found are completely covariant on the worldvolume and a complete solution is given in order to show that they are not empty (in the previous publications [5] and [12] only a partial analysis was performed). Finally, we show that, quite remarkably, as we already anticipated, this term is unique and related to the basic cohomological class of the $D = 11$ superPoincaré algebra. The application of geometrical techniques to the quantization of superstrings were already performed in series of works [14] culminating with a reformulation of the pure spinor formalism in terms of WZW model where the coset is gauged by means of the pure spinor BRST charge. A similar analysis was done for the D-branes in [15].

The paper is organized as follows: in section 2, we review some basic facts about the classical action, the κ -symmetry, the first-order formalism and the so named double first order formalism [27], which is obligatory in order to include world-volume gauge fields so as to obtain κ -symmetric actions of the Born-Infeld type and is based on the extension of the $SO(1, d)$ Lorentz symmetry to a local $GL(d, \mathbb{R})$ symmetry. This point will be relevant in the extension of our pure spinor geometrical quantization from the case of the $M2$ -brane to the case of Dp -branes. In section 3, we discuss the BRST quantization of 11d supergravity, the rheonomic parametrization of its curvatures, and the ensuing BRST symmetry with the pure spinor constraints. In section 4, the gauge fixing and the complete BRST is constructed. The nilpotency is discussed and the pure spinor constraints are solved in a well-adapted basis. In section 5, the membrane action constructed on a generic background is exemplified

on two instances of specific backgrounds. Section 6 is devoted to the conclusions and to new developments. In Appendix A we recall some definitions of the Free Differential Algebras.

2 The classical supermembrane

In the context of superstrings and of other p -brane classical theories the important issue is to write world-volume actions that possess both reparametrization invariance and κ -symmetry [2]. The former is needed to remove the unphysical degrees of freedom of the bosonic sector, while the latter removes the unphysical fermions. In this way we end up with an equal number of physical bosons and physical fermions as it is required by supersymmetry. As widely discussed in the literature [19, 20, 21, 22] the appropriate κ -symmetry transformation rules are nothing else but the supersymmetry transformation rules of the bulk supergravity background fields with a special supersymmetry parameter ϵ that is projected onto the brane. For those κ -supersymmetric branes where the gauge field strength $F_{\mu\nu}$ is not required (for example the string itself or the supermembrane) such a projection is realized by imposing that the spinor ϵ satisfies the following condition:

$$\epsilon = \frac{1}{2} \left(1 + (i)^d \frac{1}{d!} \Gamma_{\underline{a_1 \dots a_d}} V_{i_1}^{a_1} \dots V_{i_d}^{a_d} \epsilon^{i_1 \dots i_d} \right) \epsilon \quad (2.1)$$

where $\Gamma_{\underline{a}}$ are the gamma matrices in D -dimensions and V_m^a are the component of the bulk vielbein V^a onto a basis of world-volume vielbein e^m . Explicitly we write

$$V_m^b e^m = \varphi^* [V^b] \quad (2.2)$$

where $\varphi^* [V^b]$ denotes the pull-back of the bulk vielbein on the worldvolume,

$$\varphi : W_d \hookrightarrow \mathcal{M}_D \quad (2.3)$$

being the injection map of the worldvolume W_d into the target space \mathcal{M}_D .

It was shown in [20] and explicitly applied to the case of the supermembrane in [21] that by using a first order formalism on the worldvolume the implementation of κ -symmetry is reduced to an almost trivial matter once the rheonomic parametrizations, consistent with superspace Bianchi identities, are given for all the curvatures of the bulk background fields.

2.1 The first order form of the kinetic term for a p -brane

The first order formulation of the Nambu-Goto action[23] is the Polyakov action [24] for p -branes:

$$\mathcal{L}_{p\text{-brane}}^{\text{Polyakov}} = \frac{1}{2(d-1)} \int d^d \xi \sqrt{-\det h_{\mu\nu}} \left\{ h^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu g_{\underline{\mu\nu}} + (d-2) \right\} \quad (2.4)$$

where the auxiliary field $h_{\rho\sigma}$ denotes the world-volume metric. Varying the action (2.4) with respect to $\delta h_{\rho\sigma}$ we obtain the equation:

$$h_{\rho\sigma} = G_{\rho\sigma} \equiv \partial_\rho X^\mu \partial_\sigma X^\nu g_{\underline{\mu\nu}} \quad (2.5)$$

and substituting (2.5) back into (2.4) we retrieve the second order Nambu Goto action.

The Polyakov action (2.4) is not yet in a suitable form for a simple geometric implementation of κ -symmetry, but can be easily converted to such a form. The required steps are:

1. replacing the world-volume metric $h_{\mu\nu}(\xi)$ with a world-volume vielbein $e^i = e^i_\rho d\xi^\rho$,
2. using a first order formalism also for the derivatives of target space coordinates X^μ with respect to the worldvolume coordinates ξ^ρ ,
3. writing everything only in terms of flat components both on the worldvolume and in the target space.

This program is achieved by introducing an auxiliary 0-form field $\Pi^{\underline{a}}_i(\xi)$ with an index \underline{a} running in the vector representation of $\text{SO}(1, D-1)$ and a second index i running in the vector representation of $\text{SO}(1, d-1)$ and writing the action:

$$\begin{aligned} \mathcal{A}^{kin}[d] = & \int_{w_d} \left[\Pi^{\underline{a}}_j V^{\underline{b}} \eta_{\underline{ab}} \wedge \eta^{ji_1} e^{i_2} \wedge \dots \wedge e^{i_d} \epsilon_{i_1 \dots i_d} \right. \\ & \left. - \frac{1}{2d} \left(\Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta^{ij} \eta_{\underline{ab}} + d-2 \right) e^{i_1} \wedge \dots \wedge e^{i_d} \epsilon_{i_1 \dots i_d} \right] \end{aligned} \quad (2.6)$$

The variation of (2.6) with respect to $\delta \Pi^{\underline{a}}_j$ yields an equation that admits the unique algebraic solution:

$$V^{\underline{a}}|_{w_d} = \Pi^{\underline{a}}_i e^i \quad (2.7)$$

Hence the 0-form $\Pi^{\underline{a}}_i$ is identified with the intrinsic components along the world-volume vielbein e^i of the bulk vielbein $V^{\underline{a}}$ pulled-back onto the world volume. In other words the field $\Pi^{\underline{a}}_i$ is identified by its own field equation with the field $V^{\underline{a}}_i$ defined in eq. (2.2). On the other hand with the chosen numerical coefficients the variation of (2.6) with respect to the world-volume vielbein δe^i yields another equation with the unique algebraic solution:

$$\Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta_{\underline{ab}} = \eta_{ij} \quad (2.8)$$

which is the flat index transcription of eq.(2.5) identifying the world-volume metric with the pull-back of the bulk metric. Hence eliminating all the auxiliary fields via their own equation of motion the first order action (2.6) becomes proportional to the second order Nambu-Goto action. The first order form (2.6) of the kinetic action is the best suited one to discuss κ -symmetry. We consider the case of the supermembrane

2.2 κ -symmetry

In the case of the supermembrane in eleven dimensions the world-volume is three dimensional and the complete action is simply given by the kinetic action (2.6) with $d = 3$ plus the Wess-Zumino term, namely the integral of the 3-form gauge field $A^{[3]}$. Explicitly we have:

$$\mathcal{A}_{M2} = \mathcal{A}^{kin}[d=3] - \mathbf{q} \int_{w_3} A^{[3]} \quad (2.9)$$

where $\mathbf{q} = \pm 1$ is the charge of the supermembrane. As explained in [21], the background fields, namely the bulk elfbein V^a and the bulk three-form $A^{[3]}$ are superspace differential forms which are assumed to satisfy the Bianchi consistent rheonomic parametrizations of $D = 11$ supergravity as originally given in [25, 26]. Hence, although implicitly, the action functional (2.9) depends both on 11 bosonic fields, namely the $X^\mu(\xi)$ coordinates of bulk space-time, and on 32 fermionic fields $\theta^\alpha(\xi)$, forming an 11-dimensional Majorana spinor. A supersymmetry variation of the background fields is determined by the rheonomic parametrization of the curvatures and has the following explicit form:

$$\begin{aligned}\delta V^a &= i\bar{\epsilon}\Gamma^a\Psi, \\ \delta\Psi &= \mathcal{D}\epsilon - \frac{i}{3}\left(\Gamma^{\underline{b}_1\underline{b}_2\underline{b}_3}F_{\underline{a}\underline{b}_1\underline{b}_2\underline{b}_3} - \frac{1}{8}\Gamma_{\underline{a}\underline{b}_1\dots\underline{b}_4}F^{\underline{b}_1\dots\underline{b}_4}\right)\epsilon V^a, \\ \delta A^{[3]} &= -\bar{\epsilon}\Gamma^{\underline{a}\underline{b}}\Psi \wedge V_{\underline{a}} \wedge V_{\underline{b}}\end{aligned}\tag{2.10}$$

$$\tag{2.11}$$

where Ψ is the gravitino 1-form, $F_{\underline{a}_1,\dots,\underline{a}_4}$ are the intrinsic components of the $A^{[3]}$ curvature and ϵ is a 32-component spinor parameter. Essentially a supersymmetry transformation is a translation of the fermionic coordinates $\theta \mapsto \theta + \epsilon$. With such an information the κ -symmetry invariance of the action (2.9) can be established through a two-line computation, using the so called 1.5-order formalism. Technically this consists of the following: in the action (2.9) we vary only the background fields V^a and $A^{[3]}$ with respect to the supersymmetry transformations (2.11) and, after variation, we use the first order field equations (2.7) and (2.8). The action is supersymmetric if all terms that are proportional to the gravitino 1-form Ψ cancel against each other. This does not happen for a generic 32-component spinor ϵ , but it does if the latter is of the form:

$$\begin{aligned}\epsilon &= \frac{1}{2}\left(1 + i\mathbf{q}\hat{\Gamma}\right)\kappa, \\ \hat{\Gamma} &\equiv \frac{\epsilon^{ijk}}{3!}\Gamma_{ijk} = \frac{\epsilon^{ijk}}{3!}\Pi_i^{\underline{a}}\Pi_j^{\underline{b}}\Pi_k^{\underline{c}}\Gamma_{\underline{a}\underline{b}\underline{c}},\end{aligned}\tag{2.12}$$

for a generic spinor κ . Eq.(2.12) corresponds to projection (2.1) which halves the spinor components. It follows that of the 32 fermionic degrees of freedom 16 can be gauged away by κ -symmetry. The remaining 16 are further reduced to 8 by their field equations. As one sees, once the supermembrane action is cast into the first order form (2.9), κ -symmetry invariance can be implemented in an extremely simple and elegant way that requires only a couple of algebraic manipulations with gamma matrices.

2.3 Extension to $\text{GL}(3, \mathbb{R})$ invariance

The classical action (2.9) possesses the following invariances:

1. $3d$ - diffeomorphism invariance since it is written in terms of differential forms and exterior products.
2. Local $\text{SO}(1, 2)$ Lorentz invariance.
3. κ -symmetry invariance as described above.

It was noted in paper [27] that the $SO(1, d-1)$ invariance of the classical p -brane actions can be promoted to a larger $GL(d, \mathbb{R})$ invariance by introducing an additional auxiliary symmetric field h_{ij} . In this way one retrieves a supermembrane lagrangian in a set-up where the κ -symmetry is easily derived in an arbitrary supergravity background, avoiding all of its complications inherent to the second order formalism.

The $GL(3, \mathbb{R})$ -covariant classical action which replaces eq.(2.9) can be written as follows:

$$\begin{aligned}
\mathcal{A}_{class} &= \mathcal{A}_{kin} + \mathcal{A}_{WZ} \\
\mathcal{A}_{kin} &= \int \left\{ \Pi_i^a V^{\underline{b}} h^{ij} \eta_{\underline{ab}} \wedge e^j \wedge e^k \epsilon_{ijk} \right. \\
&\quad \left. - \frac{1}{6} \left[\Pi_{\underline{a}}^\ell \Pi_{\underline{b}}^m \eta^{\underline{ab}} h_{\ell m} + (\det h) \right] e^i \wedge e^j \wedge e^k \epsilon_{ijk} \right\} \\
\mathcal{A}_{WZ} &= -\mathbf{q} \int A^{[3]} \quad ; \quad \mathbf{q} = \pm 1
\end{aligned} \tag{2.13}$$

where the choice of the charge sign $\mathbf{q} = \pm 1$ corresponds to the brane/antibrane respectively.

In equation (2.13) the world-volume flat indices i, j, k are raised and lowered with the flat metric $\eta_{ij} = \text{diag}(+, -, -)$ while the Lorentz target space indices, spanning the vector representation of $SO(1, 10)$ are raised and lowered with $\eta_{\underline{ab}} = \text{diag}(+, -, \dots, -)$.

The $GL(3, \mathbb{R})$ symmetry is realized as follows. $\forall K \in GL(3, \mathbb{R})$, namely for all non-degenerate 3×3 matrix K^i_j , the following transformations:

$$\begin{aligned}
e^i &\mapsto K^i_j e^j \\
\Pi_i^a &\mapsto \Pi_\ell^a (K^{-1})^\ell_i \\
h^{ij} &\mapsto K^i_\ell K^j_m h^{\ell m} (\det K)^{-1}
\end{aligned} \tag{2.14}$$

leave the action (2.13) invariant. The transition to the second order formalism is achieved through the implementation of the field equations for the first order fields, namely Π_j^a, h_{ij} and the dreibein e^i . Let us discuss these equations one by one.

$\delta \Pi_\ell^a - \mathbf{eq.}$) Setting to zero this variation of the action (2.13) we obtain:

$$h^{\ell i} \left(V^{\underline{a}} \wedge e^j \wedge e^k \epsilon_{ijk} - \frac{1}{3} \Pi_i^{\underline{a}} \text{Vol}(3) \right) = 0 \tag{2.15}$$

where, by definition, $\text{Vol}(3) = \epsilon_{\ell_1 \ell_2 \ell_3} e^{\ell_1} \wedge e^{\ell_2} \wedge e^{\ell_3}$. Eq. (2.15) immediately implies:

$$V^{\underline{a}} = \Pi_i^{\underline{a}} e^i \tag{2.16}$$

so that the auxiliary fields $\Pi_i^{\underline{a}}$ are interpreted, once on shell, as the components of the pull-back of the bulk supergravity vielbein $V^{\underline{a}}$ onto the worldvolume of the M2-brane.

$\delta h^{ij} - \mathbf{eq.}$) Let us define the following 3×3 matrix:

$$\gamma_{ij} \equiv (\hat{\gamma})_{ij} \equiv \Pi_i^{\underline{a}} \Pi_j^{\underline{b}} \eta_{\underline{ab}} \tag{2.17}$$

In terms of this matrix the considered variation yields the following matrix equation:

$$\hat{\gamma} = \hat{h}^{-1} \left(\det \hat{h} \right) \tag{2.18}$$

where we have used the standard convention:

$$h_{ij} \equiv \left(\widehat{h}^{-1}\right)_{ij} \quad ; \quad h^{ij} \equiv \left(\widehat{h}\right)^{ij} \quad \Rightarrow \quad h_{i\ell} h^{\ell j} = \delta_i^j \quad (2.19)$$

Eq.(2.18) admits the unique solution:

$$\widehat{h} = \widehat{\gamma}^{-1} (\det \widehat{\gamma})^{1/2} \quad (2.20)$$

The interpretation of these equations is quite obvious. On shell the matrix $\widehat{\gamma}$ is the pull-back of the bulk metric onto the M2 worldvolume, written in flat components with respect to a fiducial dreibein e^i . The auxiliary field h^{ij} is just the inverse of this metric, rescaled by the square root of its determinant.

δe^k – **eq.**) The variation of the classical action (2.13) with respect to the dreibein δe^k yields the following 2-form equation:

$$2 h^{i\ell} \Pi_\ell^a V^a \wedge e^j \epsilon_{ijk} - \frac{1}{2} \left[\text{Tr} \left(\widehat{\gamma} \widehat{h} \right) + \left(\det \widehat{h} \right) \right] e^i \wedge e^j \epsilon_{ijk} = 0 \quad (2.21)$$

which is immediately translated into the following matrix equation:

$$- 2 \widehat{\gamma} \widehat{h} + \mathbf{1} \left[\text{Tr}(\widehat{h} \widehat{\gamma}) - \left(\det \widehat{h} \right) \right] = 0 \quad (2.22)$$

If we insert the solution (2.20) for \widehat{h} in terms of $\widehat{\gamma}$ into eq.(2.22) we find that it is identically satisfied. This means that in this formulation the dreibein equation imposes no new constraints once those for Π and h have been implemented. Such a feature was already stressed in the original paper [27].

The $\text{GL}(3, \mathbb{R})$ invariance of the classical action can now be used to impose suitable gauges. For instance we always have enough $\text{GL}(3, \mathbb{R})$ parameters to impose:

$$\widehat{h} = \eta \Leftrightarrow \widehat{\gamma} = \eta \quad (2.23)$$

In the gauge (2.23) eq.(2.16) reduces to eq.(2.7) and we retrieve the first order formulation of the classical action without $\text{GL}(3, \mathbb{R})$ invariance. The second order action is in any case the same, irrespectively whether we start from the old or from the new first order formalism.

It is now our program to perform a BRST quantization of the above classical supermembrane action in presence of constrained ghost fields (pure spinors). This involves three steps. First we ought to discuss the relevant BRST algebra, secondly we have to introduce a suitable gauge fixing term, thirdly we have to verify the BRST invariance of the complete quantum action and to check the nilpotency of the BRST operator. In the next section we turn to consideration of the first of these three steps.

3 BRST Quantization of $D = 11$ supergravity

The starting point for the covariant quantization of the supermembrane with pure spinors is the BRST-quantization of supergravity itself. Indeed the general strategy of the Berkovits approach consists of *constraining* some of the *ghost fields* in order to *relax* some of the *gauge degrees of freedom*. Hence as a preliminary step we have to write the complete BRST algebra of supergravity theory which includes the ghosts for all the relevant symmetries, namely:

1. $D = 11$ diffeomorphisms
2. $\text{SO}(1, 10)$ Lorentz rotations
3. 32-component local supersymmetries
4. gauge transformations of the $\mathbf{A}^{[3]}$ form.

Successively we look for a consistent set of constraints on the ghost fields which includes the complete relaxation of diffeomorphisms. This set of constraints leads to pure spinor constraints on the local supersymmetry parameters.

So let us start with the first step by deriving the BRST algebra of $D = 11$ supergravity. We follow a general procedure which was developed in [33] by extending ideas originally introduced in [34].

3.1 Rheonomy, ghost-forms and 11d supergravity

The main idea of [33] was the extension of super differential forms to generalized ghost-form (these are the generalized forms obtained from a n -form by adding a set of ghost-forms with ghost number p and form degree $n - p$ where $p = 1, \dots, n$:

$$\Omega^{[n]} \rightarrow \sum_{p=0}^n \Omega^{[n-p,p]}, \quad \Omega^{[n]} \equiv \Omega^{[n,0]}, \quad (3.1)$$

the original n -form has ghost number zero) starting from the *unique rheonomic parametrization* of superspace curvatures which defines classical supergravity theory as reviewed in the introduction. We can condensate the approach of [33] into the principle:

Principle 3.1

The correct BRST algebra is provided by replacing, in the rheonomic parametrization, of the classical supergravity curvatures each differential form with its extended ghost-form counterpart while keeping the curvature components untouched. Thus one obtains the rheonomic parametrization of the ghost-extended curvatures, whose formal definition is identical with that of the classical curvatures upon the replacements:

$$\begin{aligned} d &\mapsto d + \mathcal{S} \\ \Omega^{[n]} &\mapsto \sum_{p=0}^n \Omega^{[n-p,p]} \end{aligned} \quad (3.2)$$

In 11d supergravity, the supersymmetry transformations (2.10) which we used to define the κ -symmetry of the action (2.9) are just the consequence of the rheonomic parametrizations of the curvatures for the underlying algebraic structure of $D = 11$ supergravity. This latter is not an ordinary Lie superalgebra rather it is a Free Differential Algebra (see Appendix A for further informations). This means that the list of generators of the algebra includes, besides a set of 1-forms, spanning the dual of an ordinary Lie superalgebra \mathbb{G} , also some higher degree forms. In the specific case of $D = 11$ supergravity \mathbb{G} is just the $D = 11$ Poincaré superalgebra spanned by the following 1-forms:

1. the vielbein V^a
2. the spin connection ω^{ab}
3. the gravitino Ψ

In its minimal formulation suitable to describe the M2 brane, the relevant FDA includes just one higher degree generator namely:

- the bosonic 3-form $\mathbf{A}^{[3]}$

The complete set of curvatures describing the FDA structure is given below ([25, 26]):

$$\begin{aligned}
T^a &= \mathcal{D}V^a - i\frac{1}{2}\overline{\Psi} \wedge \Gamma^a \Psi \\
R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \\
\rho &= \mathcal{D}\Psi \equiv d\Psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab} \Psi \\
\mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} - \frac{1}{2}\overline{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b
\end{aligned} \tag{3.3}$$

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities, which for brevity we do not explicitly write (see [26]). The dynamical theory is defined, according to a general constructive scheme of supersymmetric theories, by the principle of rheonomy (see [20]) implemented into Bianchi identities. Indeed there is a unique rheonomic parametrization of the curvatures (3.3) which solves the Bianchi identities and it is the following one:

$$\begin{aligned}
T^a &= 0 \\
\mathbf{F}^{[4]} &= F_{a_1 \dots a_4} V^{a_1} \wedge \dots \wedge V^{a_4} \\
\rho &= \rho_{a_1 a_2} V^{a_1} \wedge V^{a_2} - i\frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \Psi \wedge V^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \Psi \wedge V^m \right) F^{a_1 \dots a_4} \\
R^{ab} &= R_{cd}^{ab} V^c \wedge V^d + i\rho_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \Psi \wedge V^c \\
&\quad + \overline{\Psi} \wedge \Gamma^{mn} \Psi F^{mnab} + \frac{1}{24} \overline{\Psi} \wedge \Gamma^{abc_1 \dots c_4} \Psi F^{c_1 \dots c_4}
\end{aligned} \tag{3.4}$$

The expressions (3.4) satisfy the Bianchi identities provided the space-time components of the curvatures satisfy the following constraints

$$\begin{aligned}
0 &= \mathcal{D}_m F^{mc_1 c_2 c_3} + \frac{1}{96} \epsilon^{c_1 c_2 c_3 a_1 a_8} F_{a_1 \dots a_4} F_{a_5 \dots a_8} \\
0 &= \Gamma^{abc} \rho_{bc} \\
R_{cm}^{am} &= 6 F^{ac_1 c_2 c_3} F^{bc_1 c_2 c_3} - \frac{1}{2} \delta_b^a F^{c_1 \dots c_4} F^{c_1 \dots c_4}
\end{aligned} \tag{3.5}$$

which are the space-time field equations.

At this stage the extension to ghost-forms becomes very easy. To the $(0, 1)$ -component of each element of the cotangent basis we give a specific name which will be useful for its later interpretation in the covariant quantization of the supermembrane. Explicitly we set

$$\begin{aligned}
V^a &\Rightarrow V^a + \xi^a \\
\Psi &\Rightarrow \Psi + \lambda \\
\omega^{ab} &\Rightarrow \omega^{ab} + \epsilon^{ab} \\
A^{[3]} &\Rightarrow A^{[3]} + \sum_{i=1}^3 c^{[3-i, i]}
\end{aligned} \tag{3.6}$$

where $c^{[3-i,i]}$ are $3-i$ -forms of ghost number i .

By implementing principle 3.1 and using both the definition (3.3) and the classical rheonomic parametrization of the FDA curvatures (3.4) we obtain the BRST algebra, namely the BRST transformations of all the ghost and physical fields. Explicitly, in the highest ghost number sector we find:

$$\begin{aligned}
s \xi^a - \epsilon^{ab} \xi_b &= \frac{i}{2} \bar{\lambda} \Gamma^a \lambda \\
s \epsilon^{ab} - \epsilon^{ac} \epsilon^{cb} &= R^{ab}{}_{mn} \xi^m \xi^n \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \lambda \xi^c \\
&\quad + \bar{\lambda} \Gamma^{mn} \lambda F^{mnab} + \frac{1}{24} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \lambda F^{c_1 \dots c_4} \\
s \lambda - \frac{1}{4} \epsilon^{ab} \Gamma_{ab} \lambda &= \rho_{a_1 a_2} \xi^{a_1} \xi^{a_2} \\
&\quad - i \frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \lambda \xi^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \lambda \xi^m \right) F^{a_1 \dots a_4} \\
s \mathbf{c}^{[0,3]} &= \frac{1}{2} \bar{\lambda} \Gamma_{ab} \lambda \xi^a \xi^b + F_{a_1 \dots a_4} \xi^{a_1} \dots \xi^{a_4} \\
s \mathbf{c}^{[1,2]} + d \mathbf{c}^{[0,3]} &= \bar{\lambda} \Gamma_{ab} \Psi \xi^a \xi^b + \bar{\lambda} \Gamma_{ab} \lambda \xi^a V^b \\
&\quad + 4 F_{a_1 \dots a_4} \xi^{a_1} \dots V^{a_4} \\
s \mathbf{c}^{[2,1]} + d \mathbf{c}^{[1,2]} &= \bar{\lambda} \Gamma_{ab} \lambda V^a \wedge V^b + \bar{\Psi} \wedge \Gamma_{ab} \Psi \xi^a \xi^b \\
&\quad + 2 \bar{\Psi} \wedge \Gamma_{ab} \lambda \xi^a V^b \\
&\quad + 6 F_{a_1 \dots a_4} \xi^{a_1} \dots V^{a_3} \wedge V^{a_4}
\end{aligned} \tag{3.7}$$

The next bit of information to be extracted from the quantum rheonomic parametrizations are the the BRST transformations of the physical fields, yet prior to that it is convenient to introduce a Lorentz covariant formalism by splitting the ghost extended Lorentz covariant derivative in the following way:

$$\begin{aligned}
\hat{D} &= \hat{d} + \hat{\omega}^{ab} J_{ab} \\
&= d + s + \omega^{ab} J_{ab} + \epsilon^{ab} J_{ab} \\
&= \mathcal{D} + \mathcal{S}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D} &= d + \omega^{ab} J_{ab} \quad \text{Lorentz covariant external derivative} \\
\mathcal{S} &= s + \epsilon^{ab} J_{ab} \quad \text{Lorentz covariant BRST variation}
\end{aligned} \tag{3.8}$$

where J_{ab} denotes the standard generators of the $\text{SO}(1, 10)$ Lie algebra.

The quantum rheonomic parametrization of the curvatures implies that the above operators satisfy the following algebra:

$$\begin{aligned}
\mathcal{S}^2 &= [R_{mn}^{ab} \xi^m \xi^n \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \lambda \xi^c \\
&\quad + \bar{\lambda} \Gamma^{mn} \lambda F^{mnab} + \frac{1}{24} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \lambda F^{c_1 \dots c_4}] J_{ab} \\
\mathcal{D}^2 &= [R_{mn}^{ab} V^m \wedge V^n \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \Psi \wedge V^c \\
&\quad + \bar{\Psi} \wedge \Gamma^{mn} \Psi F^{mnab} + \frac{1}{24} \bar{\Psi} \wedge \Gamma^{abc_1 \dots c_4} \Psi F^{c_1 \dots c_4}] J_{ab} \tag{3.9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S} \mathcal{D} + \mathcal{D} \mathcal{S} &= [2 R_{mn}^{ab} V^m \xi^n \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \lambda \wedge V^c \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \Psi \wedge \xi^c \\
&\quad + 2 \bar{\lambda} \Gamma^{mn} \Psi F^{mnab} + \frac{1}{12} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \Psi F^{c_1 \dots c_4}] J_{ab}
\end{aligned}$$

Utilizing these Lorentz covariant operators we can now write the BRST transformation of the physical supergravity fields as they follow from the quantum rheonomic parametrizations of the ghost-extended curvatures. We find:

$$\begin{aligned}
\mathcal{S} V^a &= -\mathcal{D} \xi^a + i \bar{\Psi} \Gamma^a \lambda \\
\mathcal{S} \Psi &= -\mathcal{D} \lambda + 2 \rho_{a_1 a_2} V^{a_1} \wedge \xi^{a_2} \\
&\quad - i \frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \lambda V^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \lambda V^m \right) F^{a_1 \dots a_4} \\
&\quad - i \frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \Psi \xi^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \Psi \xi^m \right) F^{a_1 \dots a_4} \\
\mathcal{S} \omega^{ab} &= -\mathcal{D} \epsilon^{ab} + 2 R_{mn}^{ab} V^m \xi^n \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \lambda V^c \\
&\quad + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \Psi \xi^c \\
&\quad + 2 \bar{\lambda} \Gamma^{mn} \Psi F^{mnab} + \frac{1}{12} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \Psi F^{c_1 \dots c_4} \\
\mathcal{S} \mathbf{A}^{[3]} &= -d\mathbf{c}^{[2,1]} + \bar{\Psi} \wedge \Gamma_{ab} \lambda \wedge V^a \wedge V^b \\
&\quad + \bar{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge \xi^b + 4 F_{a_1 \dots a_4} V^{a_1} \dots \xi^{a_4} \tag{3.10}
\end{aligned}$$

This concludes our presentation of the unconstrained BRST algebra of $D = 11$ supergravity. In the next section we discuss its constrained version.

3.2 Constrained BRST algebra

In agreement with the general philosophy of the pure spinor approach we reconsider the BRST algebra presented in the previous section setting to zero the parameters of diffeomorphisms, namely:

$$\xi^a \simeq 0 \quad (3.11)$$

If we set such a constraint, the BRST transformation algebra on the ghosts is easily read off from eq.s (3.7) that become:

$$\begin{aligned} 0 &= \bar{\lambda} \Gamma^a \lambda \\ \mathcal{S} \epsilon^{ab} &= \bar{\lambda} \Gamma^{mn} \lambda F^{mnab} + \frac{1}{24} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \lambda F^{c_1 \dots c_4} \\ \mathcal{S} \lambda &= 0 \\ \mathcal{S} c^{[2,1]} &= \bar{\lambda} \Gamma^{mn} \lambda V_m \wedge V_n \end{aligned} \quad (3.12)$$

At the same time when $\xi^a \simeq 0$ the BRST transformations of the physical fields (3.10) become:

$$\begin{aligned} \mathcal{S} V^a &= i \bar{\Psi} \Gamma^a \lambda \\ \mathcal{S} \omega^{ab} &= -\mathcal{D} \epsilon^{ab} + i \bar{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \lambda V^c \\ &\quad + 2 \bar{\lambda} \Gamma^{mn} \Psi F^{mnab} + \frac{1}{12} \bar{\lambda} \Gamma^{abc_1 \dots c_4} \Psi F^{c_1 \dots c_4} \\ \mathcal{S} \Psi &= -\mathcal{D} \lambda - i \frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \lambda V^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \lambda V^m \right) F^{a_1 \dots a_4} \\ &\equiv -\nabla \lambda \\ \mathcal{S} \mathbf{A}^{[3]} &= -dc^{[2,1]} + \bar{\Psi} \wedge \Gamma_{ab} \lambda \wedge V^a \wedge V^b \end{aligned} \quad (3.13)$$

- a) The first equation in (3.12) is the pure spinor constraint of the 11-dimensional superparticle.
- b) From the second of eq.s (3.12) we deduce that for flat space there are no further constraints by removing also the Lorentz ghosts
- c) The last equation in (3.12) shows that if one removes also the $c^{[2,1]}$ ghost, namely that associated with the three form gauge transformation, one more constraint on the spinor lambda pops up. Such a constraint was already seen in the literature [5]. Indeed if we want to reinstall the gauge invariance of the three form we have to set $c^{[2,1]} \approx 0$ and closure of the BRST operator implies:

$$\bar{\lambda} \Gamma_{\underline{ab}} \lambda V^{\underline{a}} \wedge V^{\underline{b}} = 0 \quad (3.14)$$

Let us now consider the pull-back of the constraint (3.14) on the worldvolume of the M2-brane. Using the gauge (2.23) we immediately see that eq.(3.14) is just equivalent to writing:

$$\bar{\lambda} \Gamma_{\underline{ab}} \lambda \Pi_{[i}^{\underline{a}} \Pi_{j]}^{\underline{b}} = 0 \quad (3.15)$$

where we have reinstalled the underlined notation $\underline{a}, \underline{b}, \underline{c}, \dots$ for the $\text{SO}(1, 10)$ vector indices, which, in the discussion of bulk supergravity, we had suppressed since no other type of

indices was needed. Here, coming back to the brane we have to distinguish target space from worldvolume indices.

It is obvious that the constraint (3.15) is certainly satisfied if we enforce the stronger one:

$$\bar{\lambda} \Gamma_{\underline{ab}} \lambda \Pi_j^b = 0 \quad (3.16)$$

found by Berkovits [5] in his hamiltonian formulation of the quantum M2 brane á la pure spinors in a flat superspace background.

In the next section we show that the first of eq.s (3.12) and (3.16) are sufficient to write a Lagrangian for the supermembrane which is BRST invariant on an arbitrary $D = 11$ supergravity background.

4 The gauge fixing and its cohomological meaning

4.1 Gauge fixing and Dirac equation

Our next task is constructing an appropriate *gauge fixing term* to be added in the usual way to the classical action, namely by writing:

$$\begin{aligned} \mathcal{A}_{class} &\mapsto \mathcal{A}_{quantum} \equiv \mathcal{A}_{class} + \mathcal{A}_{GF} \\ \mathcal{A}_{GF} &= \mathcal{S} \int \Phi_{gauge} \end{aligned} \quad (4.1)$$

where \mathcal{S} denotes the BRST operator discussed in section 3.2 and the *gauge fermion* has the standard structure:

$$\Phi_{gauge} = \text{antighost} \times \text{gauge fixing}$$

Assuming, as it usual, that:

$$\mathcal{S}(\text{antighost}) = \text{Lagrange multiplier}$$

the variation in the *Lagrange multiplier* will implement the *gauge fixing condition* as a field equation of the BRST invariant *quantum action*. What is the the local symmetry to be gauge fixed? At first sight it might seem that this is κ -symmetry, under which the classical action is invariant. Unfortunately, despite several attempts no clear and definitive answers came from [16, 17, 18].

In the present case, the classical action \mathcal{A}_{class} is not invariant under BRST symmetry (indeed the κ -symmetry parameter is replaced by a pure spinor λ), but the action $\mathcal{A}_{quantum}$ is invariant since the variation of the the first term is compensated by the variation of the second term $\mathcal{S} \int \Phi_{gauge}$. This is possible only if the BRST operator is not nilpotent. This seems strange since it is exactly the requirement of the nilpotency of the BRST operator that has led to pure spinor constraints. Nevertheless, we have to recall that the pure spinor constraints are first class constraints and they generate a gauge symmetry on the conjugated fields, namely the antighosts. Therefore, the BRST operator is nilpotent modulo the gauge symmetries on the antighosts. Those symmetries are crucial for the construction of the action.

Hence we look for a gauge fixing term with the following properties:

1. It should be $GL(3, \mathbb{R})$ invariant.
2. It should not depend neither on the dreibein e^k nor on the auxiliary fields Π_i^a , and h_{ij} , so that it will not perturb the field equations of those fields and their elimination leading to the second order action.
3. It should yield a propagation equation for the fermionic coordinates θ of target superspace which we can recognize as a standard Dirac equation for spinors on the worldvolume. Notice that the equations obtained from the \mathcal{A}_{class} are the equations of motion only for half of the θ 's.
4. Imposing BRST invariance of the action with such a gauge fermion should be consistent with the nilpotency of the BRST operator up to gauge transformations, namely:

$$\mathcal{S}^2 \text{ field} = \text{gauge transformation of that field} \quad (4.2)$$

5. The available gauge transformations, apart from Lorentz symmetry, must be those generated by the two constraints:

$$0 \approx \bar{\lambda} \Gamma^a \lambda \quad (4.3)$$

$$0 \approx \bar{\lambda} \Gamma_{\underline{ab}} \lambda \Pi_j^{\underline{b}} \quad (4.4)$$

It is quite remarkable and suggestive that the answer to the above list of requirements is provided by an essentially unique gauge-fixing endowed with a profound cohomological meaning. Before entering its description and for the reader convenience we have listed in table 1 all the fields which enter our construction, specifying also their grading and representation assignments under the two local groups $SO(1, 2)$ and $SO(1, 10)$.

Let us then state that the appropriate gauge fixing is given by the following 3-form equation:

$$\Gamma_{\underline{ab}} \Psi \wedge V^{\underline{a}} \wedge V^{\underline{b}} = 0 \quad (4.5)$$

which should hold true upon pull-back on the worldvolume.

As we anticipated the choice (4.5) has a deep cohomological meaning, since the corresponding gauge fixing term added to the Lagrangian is related to the basic cohomology 4-cycle of the super-Poincaré algebra responsible for the extension of the latter to the FDA of M-theory (see eq.(3.3)) and, ultimately, to the very existence of supermembranes.

According to table 1 and eq.(2.7)¹ equation (4.5) becomes:

$$\Gamma_{ij} \psi_k \epsilon^{ijk} = 0 \quad (4.6)$$

where we have introduced the following convenient notation:

$$\Gamma_{i_1 \dots i_n} = \Gamma_{\underline{a}_1 \dots \underline{a}_n} \Pi_{i_1}^{\underline{a}_1} \dots \Pi_{i_n}^{\underline{a}_n} \quad ; \quad n \leq 3 \quad (4.7)$$

If we recall that the gravitino 1-form Ψ always begins with the differential of the fermionic superspace coordinate:

$$\Psi = d\theta + \text{more} \quad (4.8)$$

it is evident that the constraint (4.6) is a sort of Dirac equation for worldvolume spinors.

¹Implementing the $GL(3, \mathbb{R})$ gauge (2.23), or the equation of motion of the dreibein, if we prefer the formulation without h^{ij} and without $GL(3, \mathbb{R})$ invariance.

Field name	Form degree	ghost number	SO(1, 2) repr.	SO(1, 10) repr.	Phys. Role
V^a	1	0	1	11	Target space Vielbein
Ψ	1	0	1	32	Target space gravitino
e^i	1	0	3	1	worldvolume dreibein
Π_i^a	0	0	3	11	Auxiliary field
h_{ij}	0	0	$\underbrace{\mathbf{3} \otimes \mathbf{3}}_{\text{symm}}$	1	Auxiliary field
λ	0	1	1	32	susy ghost: pure spinor
w	0	-1	1	32	antighost
Δ	0	0	1	32	Lagrange multiplier
ψ_i	0	0	3	32	name of the gravitino components along the dreibein

Table 1: *List of all the fields, of their gradings and of their representation assignments.* All the fields listed above the last line of the table are true fields appearing in the lagrangian and respect to which we are supposed to vary the action. Below the line we have listed an object ψ_i which is not a true field, rather it is the name given to the components of the pull back of the gravitino Ψ when it is expanded along the dreibein. In some sense ψ_i is the fermionic counterpart of the auxiliary field Π_i^a but differently from this latter it does not appear in the action since the fermionic action is anyhow already of the first order and geometrical.

4.2 Gauge fixing and FDA

The relation with cohomology and with the structure of the Free Differential Algebra is provided by the following considerations.

Let us recall the basic Fierz identity responsible for the existence of the 4-cycle which extends the $D = 11$ super-Poincaré Lie algebra to the FDA (3.3). It is:

$$\bar{\Psi} \wedge \Gamma_{\underline{ab}} \Psi \wedge \bar{\Psi} \wedge \Gamma^a \Psi \wedge V^{\underline{b}} = 0 \quad (4.9)$$

As it is extensively discussed in the literature [25, 26], eq.(4.9) implies that the 4-form:

$$\Omega^{[4]} \equiv \bar{\Psi} \wedge \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^{\underline{b}} \quad (4.10)$$

is closed modulo superPoincaré curvatures:

$$d\Omega^{[4]} = 0 \quad \text{at } T^a = R^{\underline{ab}} = \rho = 0 \quad (4.11)$$

and this gives origin to the 3-form $A^{[3]}$ which extends the super-Poincaré algebra to an FDA and provides the missing degrees of freedom of the $D = 11$ supergravity multiplet.

Furthermore, since it naturally couples to the worldvolume of a two-dimensional object, $\mathbf{A}^{[3]}$ allows for the existence of the M2 brane.

One can immediately note that the proposed gauge fixing (4.5) is simply the variation in $\delta\overline{\Psi}$ of the 4-cycle $\Omega^{[4]}$.

Extending the differential form algebra to the algebra of differential ghost-forms:

$$\begin{aligned}\Psi &\rightarrow \mathbf{\Psi} \equiv \Psi + \lambda \\ V^a &\mapsto \mathbf{V}^a = V^a \quad \text{at } \xi^a \approx 0 \\ d &\mapsto \mathbf{d} \equiv d + \mathcal{S}\end{aligned}\tag{4.12}$$

the 4-cycle (4.10) of d -cohomology is immediately promoted to a 4-cycle of the \mathbf{d} -operator by writing:

$$\widehat{\Omega}^{[4]} = \overline{\Psi} \wedge \Gamma_{\underline{ab}} \Psi \wedge \mathbf{V}^a \wedge \mathbf{V}^b \quad ; \quad \mathbf{d}\widehat{\Omega}^{[4]} = 0 \quad \text{modulo curvatures} \tag{4.13}$$

Expanding in ghost-number, from eq.(4.13) we obtain:

$$\begin{aligned}\widehat{\Omega}^{[4]} &= \Omega^{[4,0]} + \Omega^{[3,1]} + \Omega^{[2,2]} \\ \Omega^{[4,0]} &\equiv \overline{\Psi} \wedge \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b \\ \Omega^{[3,1]} &\equiv 2\overline{\lambda} \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b \\ \Omega^{[2,2]} &\equiv \overline{\lambda} \Gamma_{\underline{ab}} \lambda V^a \wedge V^b\end{aligned}\tag{4.14}$$

and the descent equations:

$$\mathcal{S} \Omega^{[p,q-1]} + d \Omega^{[p-1,q]} = \Xi^{[p,q]} \tag{4.15}$$

where $\widehat{\Xi}^{[5]}$ is the 5-form expressing the deviation from zero of $\mathbf{d}\widehat{\Omega}^{[4]}$ in presence of curvatures. Explicitly, since the torsion T^a is always kept zero (see (3.4)) we have:

$$\widehat{\Xi}^{[5]} = -2\overline{\Psi} \Gamma_{\underline{ab}} \widehat{\rho} \wedge \mathbf{V}^a \wedge \mathbf{V}^b \tag{4.16}$$

where, in force of the rheonomic parametrizations (3.4), we have:

$$\begin{aligned}\widehat{\rho} &= \rho^{[2,0]} + \rho^{[1,1]} \\ \rho^{[2,0]} &= \rho_{\underline{ab}} V^a \wedge V^b \\ \rho^{[1,1]} &= -i\frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \lambda V^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \lambda V^m \right) F^{a_1 \dots a_4}\end{aligned}\tag{4.17}$$

Crucial for the gauge symmetries of the quantum action with the gauge fixing (4.5) is the case $p = 3$, $q = 2$ of the descent equation (4.15), which explicitly reads:

$$\mathcal{S} \left(2\overline{\lambda} \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b \right) + d \left(\overline{\lambda} \Gamma_{\underline{ab}} \lambda V^a \wedge V^b \right) = -2\overline{\lambda} \Gamma_{\underline{ab}} \rho^{[1,1]} \wedge V^a \wedge V^b \tag{4.18}$$

Recalling the definition (3.13) of the supercovariant derivative ∇ and comparing with (4.17), eq.(4.18) can be rewritten as:

$$\mathcal{S} \left(2\overline{\lambda} \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b \right) = -\nabla \left(\overline{\lambda} \Gamma_{\underline{ab}} \lambda V^a \wedge V^b \right) \tag{4.19}$$

The relevance of the above equation in relation with the gauge symmetries of the quantum action that we are going to consider is easily explained. As we have already noted, the gauge-fixing condition we want to implement is the variation with respect to $\delta\Psi$ of the 4-cycle $\Omega^{[4]}$. Naming w the antighost which, due to its negative ghost number, can be interpreted as a contraction, a useful formal way of writing the gauge fermion (4.2) is the following:

$$\Phi_{(gauge)} = \frac{1}{2} i \overline{w} \Omega^{[4]} = \overline{w} \Gamma_{\underline{ab}} \Psi \wedge V^{\underline{a}} \wedge V^{\underline{b}} \quad (4.20)$$

Then equation (4.19) guarantees that the transformation:

$$w \mapsto w + a \lambda \quad (4.21)$$

with a an arbitrary parameter of ghost number -2 is a symmetry of the quantum action (4.1), the lagrangian varying by the total derivative of a term which is also zero on the constrained surface of pure spinors.

As we are going to see in the following, the BRST invariance of the quantum action can be achieved if the antighost field w besides (4.21), possesses a gauge symmetry which is just a slight modification of the same transformation, namely:

$$w \mapsto w + a \mathcal{P}_q \lambda \quad (4.22)$$

where

$$\mathcal{P}_{(q)} \equiv \frac{1}{2} \left(1 + i \mathbf{q} \widehat{\Gamma} \right) \quad (4.23)$$

is the κ supersymmetry projector defined in eq.(2.12). This symmetry could be established via the same chain of arguments we have just pursued if the following two statements were true:

A] The Fierz identity (4.9) can be successfully modified to the following one:

$$\overline{\Psi} \mathcal{P}_{(q)} \wedge \Gamma_{\underline{ab}} \Psi \wedge \overline{\Psi} \wedge \Gamma^{\underline{a}} \Psi \wedge V^{\underline{b}} = 0 \quad (4.24)$$

B] The κ -symmetry projector is BRST invariant:

$$\mathcal{S} \mathcal{P}_{(q)} = 0 \quad (4.25)$$

Indeed, since the operator $\widehat{\Gamma}$ is actually proportional to the volume form of the supermembrane, it is certainly true that $d\mathcal{P}_{(q)} = 0$ and condition **B]** suffices to state that:

$$\mathbf{d} \mathcal{P}_{(q)} = 0 \quad (4.26)$$

then, in case condition **A]** is also true we obtain a new ghost-form 4 cycle:

$$\widehat{\Omega}_{\mathbf{q}}^{[4]} = \overline{\Psi} \mathcal{P}_{(q)} \wedge \Gamma_{\underline{ab}} \Psi \wedge \mathbf{V}^{\underline{a}} \wedge \mathbf{V}^{\underline{b}} \quad ; \quad \mathbf{d} \widehat{\Omega}_{\mathbf{q}}^{[4]} = 0 \quad \text{modulo curvatures} \quad (4.27)$$

and a new analogous chain of descent equations (4.15):

$$\mathcal{S} \Omega_{\mathbf{q}}^{[p, q-1]} + d \Omega_{\mathbf{q}}^{[p-1, q]} = \Xi_{\mathbf{q}}^{[p, q]} \quad (4.28)$$

leading in particular to the modified version of eq.(4.19):

$$\mathcal{S} \left(2 \bar{\lambda} \mathcal{P}_{(q)} \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b \right) = -\nabla \left(\bar{\lambda} \mathcal{P}_{(q)} \Gamma_{\underline{ab}} \lambda V^a \wedge V^b \right) \quad (4.29)$$

which guarantees the invariance of the quantum action (4.1) under the transformation (4.22).

It turns out that on the constrained surface of pure spinors and upon pull-back onto the membrane worldvolume, conditions **A]** and **B]** are indeed true. To prove it, we rely on the use of a well-adapted basis of gamma matrices where we are able to solve the pure spinor constraints explicitly and henceforth to derive a series of identities which, at the end of the calculation, we can recast in a fully $D = 11$ covariant form and by this token obtain the desired verification of the statements we just made. Such a derivation is discussed in the next section.

4.3 Pure spinors in a well adapted gamma basis and proof of the relevant identities

The implications of the pure spinor constraints (4.3-4.4) are quite easily discussed and solved if we refer to a gamma matrix basis which is well adapted to the splitting of the eleven dimensions in $\mathbf{3} \oplus \mathbf{8}$, the first $\mathbf{3}$ being the dimensions occupied by the M2-brane worldvolume, the remaining $\mathbf{8}$ being those transverse to the brane. According to this we write

$$\Gamma_{\underline{a}} = \begin{cases} \Gamma_i &= \gamma_i \otimes T_9 \quad ; \quad i = 0, 1, 2 \\ \Gamma_{2+A} &= \mathbf{1} \otimes T_A \quad ; \quad A = 1, 2, \dots, 8 \end{cases} \quad (4.30)$$

where γ_i are 2×2 gamma matrices for the $\text{SO}(1, 2)$ Clifford algebra, namely:

$$\{\gamma_i, \gamma_j\} = 2\eta_{ij} = \text{diag}\{+, -, -\} \quad (4.31)$$

while T_A are 16×16 gamma matrices for the $\text{SO}(8)$ Clifford algebra with negative metric:

$$\{T_A, T_B\} = -2\delta_{AB} \quad (4.32)$$

As an explicit representation of the $d = 3$ gamma matrices in presence of a mostly minus metric we can take the following ones in terms of Pauli matrices:

$$\gamma_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad ; \quad \gamma_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.33)$$

On the other hand the $\text{SO}(8)$ Clifford algebra with negative metric admits a representation in terms of completely real and antisymmetric matrices. We adopt the following one:

$$T_A = \begin{cases} T_\alpha &= \sigma_1 \otimes \tau_\alpha \quad ; \quad \alpha = 1, 2, \dots, 7 \\ T_8 &= i\sigma_2 \otimes \mathbf{1}_{8 \times 8} \quad ; \end{cases} \quad (4.34)$$

where τ_α denotes the 8×8 completely antisymmetric realization of the $\text{SO}(7)$ Clifford algebra with negative metric:

$$\{\tau_\alpha, \tau_\beta\} = -2\delta_{\alpha\beta} \quad ; \quad \tau_\alpha = -(\tau_\alpha)^T \quad (4.35)$$

given by:

$$\begin{aligned}
\tau_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad \tau_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\tau_3 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad \tau_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\tau_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} ; \quad \tau_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \\
\tau_7 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{4.36}$$

This realization of the τ matrices admits the following interpretation:

$$(\tau_\alpha)_{\beta\gamma} = a_{\alpha\beta\gamma} \quad ; \quad (\tau_\alpha)_{\beta 8} = -(\tau_\alpha)_{8\beta} = \delta_{\alpha\beta} \tag{4.37}$$

where the completely antisymmetric tensor $a_{\alpha\beta\gamma}$ encodes the structure constants of the octonion algebra or, equivalently corresponds to the components of the unique G_2 invariant 3-form.

Finally the 16×16 matrix T_9 which anticommutes with all the T_A has, in this basis, the following structure:

$$T_9 = -\sigma_3 \otimes \mathbf{1}_{8 \times 8} \tag{4.38}$$

The charge conjugation matrix, with respect to which we have:

$$C \Gamma_{\underline{a}} C^{-1} = -\Gamma_{\underline{a}}^T \tag{4.39}$$

is given by:

$$C = \varepsilon \otimes \mathbf{1}_{16 \times 16} \quad ; \quad (\varepsilon \equiv i\sigma_2) \tag{4.40}$$

Within this setup we can now address the problem of solving the pure spinor constraints (4.3-4.4). To this end we begin by parametrizing a *complex* 32-component spinor λ as the following tensor product:

$$\lambda = \phi_+ \otimes \zeta_+ + \phi_- \otimes \zeta_- \tag{4.41}$$

where $T_9 \zeta_{\pm} = \pm \zeta_{\pm}$ are $SO(8)$ spinors of opposite chiralities and ϕ^{\pm} are 2-component $SO(1, 2)$ spinors. Calculating the one-gamma current we obtain:

$$\bar{\lambda} \Gamma_{\underline{a}} \lambda \equiv \lambda^T C \Gamma_{\underline{a}} \lambda = \begin{cases} (\phi_+^T \gamma_i \phi_+) \zeta_+^T \zeta_+ - (\phi_-^T \gamma_i \phi_-) \zeta_-^T \zeta_- & ; \quad i = 0, 1, 2 \\ 2 (\phi_+^T \varepsilon \phi_-) \zeta_+^T T_A \zeta_- & ; \quad A = 1, \dots, 8 \end{cases} \quad (4.42)$$

Hence the first of the two constraints (4.3) can be easily solved by means of the following two equations:

$$\begin{aligned} \phi_+ &= \phi_- = \phi \\ \zeta_+^T \zeta_+ &= \zeta_-^T \zeta_- \end{aligned} \quad (4.43)$$

which lead to a pure spinor having 23 independent components. The argument to count 23 is the following one. By means of eq.(4.43) we have explicitly constructed a parametric solution of the pure spinor constraint equation in terms of an object which has 32 in general non-zero complex components. We have to see how many relations are imposed on these 32 components by the parametric solution. This is easily done. Let us enumerate, according to the adopted tensor product structure the components of the spinor (4.41) in the following way:

$$\begin{aligned} \lambda^I &= \phi_+^1 \zeta_-^I & ; & \quad \lambda^{I+8} = \phi_-^1 \zeta_+^I \\ \lambda^{I+16} &= \phi_+^2 \zeta_-^I & ; & \quad \lambda^{I+24} = \phi_-^2 \zeta_+^I \end{aligned} \quad (4.44)$$

It is evident that the first of the two equations (4.43) imposes exactly 8 equations on the spinor components, namely:

$$\frac{\lambda^I}{\lambda^{I+16}} = \frac{\lambda^{I+8}}{\lambda^{I+24}} \quad ; \quad I = 1, \dots, 8 \quad (4.45)$$

Hence after the first of (4.43) has been imposed, we have 24 independent components. The second of (4.43) imposes just one more condition:

$$\sum_{I=1}^{16} \left[(\lambda^I)^2 - (\lambda^{I+16})^2 \right] = 0 \quad (4.46)$$

which reduces the spinor to 23 independent components as it is well known in the literature [5].

Let us now consider the second constraint (4.4). In the following we will consider the second constraint as a primary constraints and therefore we solve it in the same spirit as (4.3). In the well-adapted gamma matrix basis this reduces to:

$$\begin{aligned} 0 &= \phi^T \varepsilon \gamma_{ij} \phi (\zeta_+^T \zeta_+ + \zeta_-^T \zeta_-) \\ 0 &= \phi^T \varepsilon \gamma_i \phi (\zeta_+^T T_A \zeta_-) \end{aligned} \quad (4.47)$$

which can be easily and uniquely solved by choosing either one of the following four positions:

$$\begin{aligned}
\zeta_- &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \quad \zeta_+ = \begin{pmatrix} 0 \\ \omega \end{pmatrix} ; \quad \omega^T \omega = 0 \\
&\text{or} \\
\zeta_+ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \quad \zeta_- = \begin{pmatrix} \omega \\ 0 \end{pmatrix} ; \quad \omega^T \omega = 0 \\
&\text{or} \\
\zeta_+ &= \begin{pmatrix} 0 \\ \omega \end{pmatrix} ; \quad \zeta_- = \begin{pmatrix} \omega \\ 0 \end{pmatrix} ; \quad \omega^T \omega = 0 \\
&\text{or} \\
\zeta_+ &= \begin{pmatrix} 0 \\ \omega \end{pmatrix} ; \quad \zeta_- = \begin{pmatrix} -\omega \\ 0 \end{pmatrix} ; \quad \omega^T \omega = 0
\end{aligned} \tag{4.48}$$

where in all four cases ω is an 8-component complex object with vanishing norm. The manifold of pure spinors has therefore four disjoint branches given by the four options in eq.(4.48). In any case the pure spinor satisfying all the constraints has 15 independent components.

In the well adapted basis the world-volume components ψ_i of the gravitino field which appear in the gauge-fixing (4.6) can be parametrized as follows:

$$\psi_i = \mu_i^+ \otimes \chi_+ + \mu_i^- \otimes \chi_- \tag{4.49}$$

where μ_i^\pm are 3×2 -component vector-spinors of $\text{SO}(1, 2)$ and χ^\pm are 16-component spinors of $\text{SO}(8)$. In this parametrization the gauge-fixing equation (4.6) reduces to:

$$(\gamma_{ij} \mu_k^+ \otimes \chi_+ - \gamma_{ij} \mu_k^- \otimes \chi_-) \epsilon^{ijk} = 0 \tag{4.50}$$

which implies:

$$\gamma_i \mu_j^\pm \eta^{ij} = 0 \tag{4.51}$$

Relying on eq.(4.51) we can now prove a Fierz identity which will be crucial in demonstrating the nilpotency of the BRST operator. The identity is the following. Define the object:

$$\Upsilon \equiv \Gamma_{\underline{abc}} \lambda \bar{\lambda} \Gamma^{\underline{a}} \psi_i \Pi_j^{\underline{b}} \Pi_k^{\underline{c}} \epsilon^{ijk} \tag{4.52}$$

where λ is the pure spinor ghost field and the other items entering the definition have already been defined above. We want to show, that independently of the choice of the branch (4.48) the structure Υ vanishes modulo the fermionic field equations, namely upon enforcement of eq.s (4.51). To this effect it suffices to split the sum on the index \underline{a} into the sum over the first three indices m and over the last eight indices A . So we write:

$$\begin{aligned}
\Upsilon &= \Upsilon_{[3]} + \Upsilon_{[8]} \\
\Upsilon_{[3]} &= \Gamma_{pjk} \lambda \bar{\lambda} \Gamma^p \psi_i \epsilon^{ijk} \\
\Upsilon_{[8]} &= \Gamma_{Ajk} \lambda \bar{\lambda} \Gamma^A \psi_i \epsilon^{ijk}
\end{aligned} \tag{4.53}$$

and in the well adapted basis we find:

$$\begin{aligned}
\Upsilon_{[3]} &= \gamma_{pjk} \phi \bar{\phi} \gamma^p \mu_i^+ \epsilon^{ijk} \otimes T_9 \zeta \zeta^T T_9 \chi_+ + \gamma_{pjk} \phi \bar{\phi} \gamma^p \mu_i^- \epsilon^{ijk} \otimes T_9 \zeta \zeta^T T_9 \chi_- \\
&= -2i \left(\phi \bar{\phi} \gamma^i \mu_i^+ \otimes T_9 \zeta \zeta^T T_9 \chi_+ + \phi \bar{\phi} \gamma^i \mu_i^- \otimes T_9 \zeta \zeta^T T_9 \chi_- \right) \\
&\approx 0 \quad \text{modulo field eq.s}
\end{aligned} \tag{4.54}$$

which is a consequence of $\gamma_{ijk} = -i \mathbf{1}_{2 \times 2} \epsilon_{ijk}$.

On the other hand for the second structure we have:

$$\Upsilon_{[8]} = \gamma_{ij} \phi \bar{\phi} \mu_k^+ \epsilon^{ijk} \otimes T_A \zeta \zeta^T T_A \chi_+ + \gamma_{ij} \phi \bar{\phi} \mu_k^- \epsilon^{ijk} \otimes T_A \zeta \zeta^T T_A \chi_- \tag{4.55}$$

Starting from eq (4.55) we can show that also $\Upsilon_{[8]}$ vanishes using the following identity:

$$T_A \zeta \zeta^T T_A = \omega^T \omega \frac{1}{2} (\mathbf{1} - T_9) = 0 \tag{4.56}$$

which is true for all four cases of spinors ζ listed in eq.(4.48). In this way, independently from the choice of the branch (4.48) in the solution of the pure spinor constraints (4.3) and (4.4) we have shown that $\Upsilon = 0$ modulo field equations.

It is interesting to rewrite in a $D = 11$ fully covariant way the result for off-shell Υ . To this end it suffices to compare the result obtained in equation (4.54) with the structure of the κ -symmetry projector (2.12) in the well-adapted basis. Using (4.30) we obtain:

$$\begin{aligned}
\mathcal{P}_{(q)} &\equiv \frac{1}{2} \left(\mathbf{1} + i \mathbf{q} \hat{\Gamma} \right) \\
&= \mathbf{1}_{2 \times 2} \otimes \frac{1}{2} (\mathbf{1}_{16 \times 16} + T_9)
\end{aligned} \tag{4.57}$$

On the other hand the second line in eq.(4.54) can be rewritten as:

$$\Upsilon_{[3]} = -2i \mathbf{1}_{2 \times 2} \otimes T_9 \lambda \bar{\lambda} \Gamma^i \psi_i \tag{4.58}$$

and in view of (4.57) and of the off-shell vanishing of $\Upsilon_{[8]}$ we can conclude that Υ as defined in eq.(4.52) is also equal to the following expression:

$$\Upsilon = 2 \hat{\Gamma} \lambda \lambda \Gamma^i \psi_i \tag{4.59}$$

Our next point is to show that in off-shell second order formalism, namely upon implementation of the algebraic field equation for the auxiliary fields Π_i^a and e^i , without imposing any constraint on the physical fields, the κ supersymmetry projector operator is BRST invariant, namely that eq.(4.25) is true off-shell. Since the pure spinor ghost is anyhow BRST invariant $\mathcal{S} \lambda = 0$, eq.(4.25) is completely equivalent to the equation below:

$$\mathcal{S} [\mathcal{P}_{(q)} \lambda] = 0 \quad (\text{off-shell}) \tag{4.60}$$

which is what we can prove by using the identity (4.59). To this effect let us anticipate a result which we derive in the next section while discussing the BRST invariance of the action. This latter requires the BRST variation of the world-volume dreibein to be of the following form:

$$\mathcal{S} e^i = \eta^{im} N_{im} e^m \tag{4.61}$$

with the condition on N to be symmetric $N_{ij} = N_{ji}$. Using the equations of motion for the auxiliary fields Π_i^a and e^i , namely using the second-order formalism, we are able to determine N_{ij} explicitly. Combining eq.(2.16) (*i.e.* the field equation of Π_i^a) with eq.(3.13) (the BRST variation of the bulk vielbein), and also eq.(4.61), we obtain:

$$\mathcal{S} \Pi_\ell^a = i \bar{\lambda} \Gamma^a \psi_\ell - \Pi_f^a \eta^{fm} N_{m\ell} \quad (4.62)$$

Considering next eq.(2.8), which is just the field equation of the dreibein in the formulation without h^{ij2} , we get:

$$0 = \mathcal{S} \eta_{ij} = \mathcal{S} \Pi_i^a \Pi_j^b \eta_{ab} + \mathcal{S} \Pi_j^a \Pi_i^b \eta_{ab} \quad (4.63)$$

From (4.63) and (4.62) we immediately obtain:

$$N_{ij} = i \bar{\lambda} \Gamma_{(i} \psi_{j)} \quad (4.64)$$

Equipped with these intermediate results we can calculate:

$$\begin{aligned} \mathcal{S} [\mathcal{P}_{(q)} \lambda] &= \frac{3}{2} \mathbf{q} \Upsilon + \frac{3}{2} i \mathbf{q} \Gamma_{abc} \lambda \Pi_f^a \Pi_f^b \Pi_r^c \eta^{fm} N_{mp} \epsilon^{pqr} \\ &= \frac{3}{2} \mathbf{q} \Upsilon + 3 i q \hat{\Gamma} \lambda (N_{pq} \eta^{pq}) \\ &= -3 i q \hat{\Gamma} \lambda (N_{pq} \eta^{pq}) + 3 i q \hat{\Gamma} \lambda (N_{pq} \eta^{pq}) = 0 \end{aligned} \quad (4.65)$$

which proves eq. (4.60) and hence also eq.(4.25) The first condition, namely the Fierz identity (4.24) can be proved in the same well adapted basis by explicit evaluation for instance on a computer. It is just an algebraic identity and it is indeed true.

4.4 Gauge fixing term and BRST invariance

Relying on the identities shown in the previous sections we can now prove that the quantum action defined by eq.(4.1), with the gauge fermion provided by eq.(4.20) is indeed BRST invariant.

According to table 1 we recall that the antighost w is just a target space spinor and we set the following BRST transformation rules:

$$\mathcal{S} w = \Delta \quad ; \quad \mathcal{S} \Delta = \vartheta \quad (4.66)$$

where ϑ is an object to be determined in such a way that the final action be BRST invariant. The gauge fermion being that in eq.(4.20) the explicit form of the gauge fixing action is easily evaluated:

$$\begin{aligned} \mathcal{A}_{GF} &= \mathcal{S} \int \bar{w} \Gamma_{ab} \psi \wedge V^a \wedge V^b \\ &= \int \{ \bar{\Delta} \Gamma_{ab} \psi \wedge V^a \wedge V^b - \bar{w} \Gamma_{ab} \nabla \lambda \wedge V^a \wedge V^b \\ &\quad + 2 i \bar{w} \Gamma_{ab} \Psi \wedge \bar{\lambda} \Gamma^a \Psi \wedge V^b \} \end{aligned} \quad (4.67)$$

²Or can be alternatively imposed as a gauge fixing condition in the formulation with $GL(3, \mathbb{R})$ symmetry

If we calculate the BRST transformation of this part of the quantum action we simply obtain:

$$\begin{aligned}
\mathcal{S} \mathcal{A}_{GF} &= \int \mathcal{S}^2 \bar{\Psi} \Gamma_{\underline{ab}} \Psi \wedge V^a \wedge V^b = \int \bar{\Psi} \Gamma_{\underline{ab}} \vartheta \wedge V^a \wedge V^b \\
&= \int \bar{\Psi} \Gamma_{ij} \vartheta \wedge e^i \wedge e^j \\
&= -\frac{1}{6} \int \bar{\psi}_k \Gamma_{ij} \vartheta \epsilon^{kij} \text{Vol}(3)
\end{aligned} \tag{4.68}$$

where we have used the notation $\text{Vol}(3) = \epsilon_{\ell_1 \ell_2 \ell_3} e^{\ell_1} \wedge e^{\ell_2} \wedge e^{\ell_3}$ already introduced before. The question is whether there exists a ϑ appropriate to cancel the BRST variation of the classical action. The idea behind this procedure is that the nilpotency of the BRST operator is preserved if eq.(4.2) is true on all fields. Hence, in view of our previous discussions, ϑ should be one of the gauge symmetries of the antighost w , namely, either ϑ should be proportional to λ or to $\mathcal{P}_q \lambda$. We will explicitly demonstrate that the second is the right choice. In both cases the results obtained in the previous sections already guarantee that:

$$\mathcal{S}^3 w = \mathcal{S} \vartheta = 0 \tag{4.69}$$

as it should be for consistency.

So let us now calculate the BRST variation of the classical action. Here we use the 1.5 order formalism, namely we vary the first order action (2.13), but we consider only the variation of the physical fields, ignoring that of the auxiliary fields. Then after variation we implement, for the auxiliary fields their value set by their own field equation. So, for the classical action we find:

$$\begin{aligned}
\mathcal{S} \mathcal{A}_{class} &= \int \left\{ i \Pi_{\underline{a}}^i \bar{\Psi} \Gamma^a \lambda \wedge e^j \wedge e^k \epsilon_{ijk} - \mathbf{q} \bar{\Psi} \Gamma_{\underline{ab}} \lambda \wedge V^a \wedge V^b \right\} \\
&= \int \left\{ i \bar{\Psi} \Gamma^i \lambda \wedge e^j \wedge e^k \epsilon_{ijk} - \mathbf{q} \bar{\Psi} \Gamma_{ij} \lambda \wedge e^i \wedge e^j \right\} \\
&= \int i \frac{2}{3} \bar{\lambda} \Gamma_i \mathcal{P}_q \psi_j \eta^{ij} \text{Vol}(3)
\end{aligned} \tag{4.70}$$

where we have used the identity:

$$\begin{aligned}
i \frac{2}{3} \eta^{ij} \bar{\lambda} \Gamma_i \mathcal{P}_q \psi_j &= i \frac{1}{3} \bar{\lambda} \Gamma_i \psi_j - \mathbf{q} \frac{1}{6} \bar{\lambda} \Gamma_{ij} \psi_m \epsilon^{ijm} \\
&= -i \frac{2}{3} \eta^{ij} \bar{\psi}_i \Gamma_j \mathcal{P}_q \lambda
\end{aligned} \tag{4.71}$$

which follows from standard gamma matrix manipulations. Similarly one can prove the other identity:

$$\frac{1}{3} \bar{\psi}_k \Gamma_{ij} \mathcal{P}_q \lambda = i \mathbf{q} \eta^{ij} \bar{\psi}_i \Gamma_j \mathcal{P}_q \lambda \tag{4.72}$$

Combining these results we conclude that it suffices to set:

$$\vartheta = 4 \mathcal{P}_q \lambda \tag{4.73}$$

which is consistent with our previous statements. Indeed it corresponds to a shift symmetry of the antighost and satisfies the closure condition (4.69)

4.5 Primary, Secondary and Pure Spinor Constraints

Before closing this section, we have to spend some words concerning the constraint structure of the theory and for that we use the Dirac procedure.

In contrast to [5], we adopt two types of constraints from the beginning: (4.3) and (4.4) since they are implied by the supersymmetry algebra. So, we consider them as primary constraints. Those two constraints are first class constraints since they commute (using the Poisson brackets)

$$\{\bar{\lambda}\Gamma^a\lambda, \bar{\lambda}\Gamma^b\lambda\} = 0, \quad \{\bar{\lambda}\Gamma^a\lambda, \bar{\lambda}\Gamma^{bc}\lambda\Pi_{\underline{ci}}\} = 0, \quad \{\bar{\lambda}\Gamma^{ad}\Pi_{\underline{aj}}\lambda, \bar{\lambda}\Gamma^{bc}\lambda\Pi_{\underline{ci}}\} = 0, \quad (4.74)$$

where we have assumed that $\{\Pi_i^a, \Pi_j^b\} = 0$ (since Π_i^b is the conjugate momentum to x^a). In addition, using the Fierz identities one can prove that they are BRST invariant. Nevertheless, the action $\mathcal{A}_{quantum}$ is non-linear and therefore the primary constraints yield secondary constraints by computing the commutator between the Hamiltonian and the primary constraints. However, since we want to stick to Lagrangian formalism, we can check whether the action is invariant under the gauge symmetries generated by the primary constraints. We found that there are secondary constraints which are differential constraints and automatically implemented by imposing the field equations. Furthermore, as we have already seen in previous sections, the only gauge symmetries which do not impose any further constraints are those which are needed to close the BRST algebra (4.69).

We have to enumerate some differences with the previous approach using the Hamiltonian formalism [5]. We have seen that the naive covariantization of the field equations for θ^α

$$\partial_0\theta^\alpha + \epsilon^{IJ}\Gamma_{\underline{\beta}}^{a\alpha}\Pi_{\underline{aI}}\partial_J\theta^\beta = 0 \quad \longrightarrow \quad \gamma^{ij}\partial_j\theta^\alpha + \epsilon^{ijk}\Gamma_{\underline{\beta}}^{a\alpha}\Pi_{\underline{aj}}\partial_k\theta^\beta = 0 \quad (4.75)$$

is not consistent. The reason is that the new equation imposes too strong constraints on the fields θ 's which has only constant solution. Therefore, we decided to use the Dirac equation for the second half of the θ 's (those which would have been projected away by the κ -symmetry) and we get a more complicate action. The way to see that the number of conjugate fields $w_{\underline{a}}$ is correct is to check that the wave operator is a quadratic matrix in spinor representation.

It is interesting to compare the supermembrane action with the superstring action quantized with the pure spinors [4]. In particular, we compare the “gauge fixing part” $\mathcal{S} \int \Phi_{gauge}$ in Hamiltonian and Lagrangian formalism (in the Hamiltonian formalism the BRST differential operator \mathcal{S} is replaced by the BRST charge Q). We recall that the gauge fixing term for the superstring (we consider here type IIA to compare with the supermembrane action) in the worldsheet light-cone coordinates reads as follows

$$\mathcal{A}_{gauge} = Q \int d^2z \left(w_{\alpha L z} \bar{\partial}\theta_L^\alpha + w_{R\bar{z}}^\alpha \partial\theta_{\alpha R} \right). \quad (4.76)$$

This part action admits two different interpretations. On one hand, we can see (4.76) as a gauge-fixed version of the following action

$$\mathcal{A}_{gauge} = \mathcal{S} \int d\sigma d\tau \left(w_L^i (\eta_{ij} + \epsilon_{ij}) d\theta_L \wedge e^j + w_R^i (\eta_{ij} - \epsilon_{ij}) d\theta_R \wedge e^j \right). \quad (4.77)$$

where the second components for $w_{\alpha Lz}$ and for $w_{R\bar{z}}^\alpha$ have been introduced. The two projectors $(\eta_{ij} \pm \epsilon_{ij})$ imply that the action (4.77) is invariant under $\delta w_{L/R}^i = (\eta_{ij} \mp \epsilon_{ij})\varphi_j$. On the other hand, we obtain the action (4.76) by dimensional reduction from the supermembrane action

$$\mathcal{A}_{gauge} = \mathcal{S} \int d^3x \left(w \epsilon^{ijk} \Pi_i^b \Pi_j^a \Gamma_{ab} \partial_k \theta \right). \quad (4.78)$$

First, we split the 11d indices \underline{a} into 10d indices $(a, 11)$ and then, we integrate over the third worldvolume coordinate. This eliminates the contribution coming from Γ_{ab} and we use the usual gamma matrix identifications $\Gamma_{11} = \gamma_3 \otimes T_9$ and $\gamma_i = \gamma_3 \epsilon_{ij} \gamma^j$. Thus, we are left with the 2-forms terms

$$\mathcal{A}_{gauge} = \mathcal{S} \int d\sigma d\tau \left(w_L \gamma_a d\theta_L - w_R \gamma_a d\theta_R \right) \wedge V^a, \quad (4.79)$$

where we have redefined the antighost fields $w \rightarrow (w_L, w_R)$ in a suitable way. It is obvious to compare this action with the string action with the pure spinors. In this way, we see that the antighosts of superstrings can also be represented by a spinor rather than by a vector spinor $w_{L/R}^i$ as it is mandatory in the supermembrane. The formulation in term of a single spinor antighost is natural from the viewpoint of FDA. Furthermore, one can also see that the pure spinor constraints (4.3)-(4.4) used in the present framework are directly related to the pure spinor constraints of superstring type IIA. We plan to discuss this point further in a future work.

5 Examples: the quantum action of the supermembrane on specific backgrounds

The construction that we have described in the previous sections provides explicit formulae for the supermembrane quantum action in terms of target space supercoordinates any time we have at our disposal an explicit parametrization of the $D = 11$ superspace geometry. All what we need are just three geometrical data:

1. The supervielbein one-form $V^{\underline{a}}$
2. The gravitino one-form Ψ
3. The three-form $A^{[3]}$

As an illustration of the ultimate content of our result we consider two explicit cases, namely flat D=11 superspace and the $\text{AdS}_4 \times \text{S}^7$ solution.

We remind the reader that the pure spinor superstring on arbitrary background has been studied in [28] where it is shown that the pure spinor conditions and the holomorphicity conditions imply the supergravity constraints for heterotic and type IIA/B superstrings. The background formulation of pure spinor supermembrane is analyzed in [5] in Hamiltonian formalism. On the other hand, the arbitrary background formulation of κ -symmetric p -branes has been initiated in [21, 30] for $M2$ and the superparticle and then extended to any p -brane including also worldvolume gauge fields in [27, 29].

5.1 $D = 11$ flat superspace

In this case the representation of the needed supergeometrical data is very simple. Naming X^a the eleven bosonic coordinates and θ the 32-component Majorana spinor, the structural equations (3.3) with zero curvatures are immediately solved by setting:

$$\begin{aligned} V^a &= dX^a + i\bar{\theta}\Gamma^a d\theta \\ \Psi &= d\theta \\ \mathbf{A}^{[3]} &= \frac{1}{2}\bar{\theta}\Gamma_{\underline{ab}}d\theta \wedge dX^a \wedge dX^b + i\frac{1}{4}\bar{\theta}\Gamma_{\underline{ab}}d\theta \wedge \bar{\theta}\Gamma^a d\theta \wedge dX^b \\ &\quad - \frac{1}{12}\bar{\theta}\Gamma_{\underline{ab}}d\theta \wedge \bar{\theta}\Gamma^a d\theta \wedge \bar{\theta}\Gamma^b d\theta \end{aligned} \quad (5.1)$$

In proving that the last line of eq.(5.1) does indeed satisfy the required relation:

$$d\mathbf{A}^{[3]} = \frac{1}{2}\bar{\Psi} \wedge \Gamma_{\underline{ab}}\Psi \wedge V^a \wedge V^b \quad (5.2)$$

one has just to rely on the following Fierz identity:

$$\bar{\theta}\Gamma_{\underline{ab}}d\theta \wedge d\bar{\theta} \wedge \Gamma^a d\theta = -d\bar{\theta} \wedge \Gamma_{\underline{ab}}d\theta \wedge \bar{\theta}\Gamma^a d\theta \quad (5.3)$$

Substituting the above formulae into the supermembrane action it becomes fully explicit in terms of all the fields.

5.2 The supermembrane on $\text{AdS}_4 \times \mathbb{S}^7$

Another interesting case of backgrounds on which the quantum action of the supermembrane can be considered is provided by the Freund-Rubin solutions of $D = 11$ supergravity of type³

$$\mathcal{M}_{11} = \text{AdS}_4 \times \frac{\text{G}}{\text{H}} \quad (5.4)$$

where AdS_4 denotes 4-dimensional anti de Sitter space and G/H is a 7-dimensional coset manifold equipped with an invariant Einstein metric and admitting \mathcal{N} Killing spinors η_A . Naming y the coordinates of such a 7-manifold its vielbein and spin-connection one-forms are respectively denoted $\mathcal{B}^\alpha(y), \mathcal{B}^{\alpha\beta}(y)$ and, in order to solve the D=11 field equations, they satisfy the following structural equations:

$$\begin{aligned} d\mathcal{B}^\alpha + \mathcal{B}^{\alpha\beta} \wedge \mathcal{B}^\gamma \eta_{\beta\gamma} &= 0 \\ d\mathcal{B}^{\alpha\beta} + \mathcal{B}^{\alpha\gamma} \wedge \mathcal{B}^{\delta\beta} \eta_{\beta\gamma} &= \mathcal{R}^{\alpha\beta}{}_{\gamma\delta} \mathcal{B}^\gamma \wedge \mathcal{B}^\delta \\ \mathcal{R}^{\alpha\beta}{}_{\gamma\delta} &= 12 e^2 \delta_\beta^\alpha \end{aligned} \quad (5.5)$$

where e is named the Freund-Rubin parameter and it is the only scale parameter of the entire solution. In terms of these objects the Killing spinors are eight component spinors of the tangent group $\text{SO}(7)$ that are required to satisfy the following equation:

$$d\eta_a + \frac{1}{4}\mathcal{B}^{\alpha\beta}\tau_{\alpha\beta}\eta_A = e\mathcal{B}^\alpha\tau_\alpha\eta_A \quad ; \quad (A = 1, \dots, \mathcal{N}) \quad (5.6)$$

³For a review of Kaluza-Klein compactifications of M-theory we refer the reader to [20]

In eq.(5.6), by τ_α we have denoted the seven dimensional 8×8 gamma matrices already introduced in section 4.3 which satisfy the standard Clifford algebra with negative metric $\eta_{\alpha\beta} = -\delta_{\alpha\beta}$. The complete symmetry of the solution (5.4) is given by the supergroup:

$$\mathcal{SG} = \text{Osp}(\mathcal{N}|4) \times \text{N}_G(\text{SO}(\mathcal{N})) \quad (5.7)$$

where $\text{N}_G(\text{SO}(\mathcal{N}))$ denotes the normalizer of the R-symmetry group $\text{SO}(\mathcal{N})$ within the full isometry group G of the internal seven manifold. In the book [20] it was shown how to construct the full $D = 11$ superspace geometry corresponding to this class of solutions in terms of the geometrical data specified above and in terms of the super geometry of the super coset manifold:

$$\mathcal{SM} = \frac{\text{Osp}(\mathcal{N}|4)}{\text{SO}(1,3) \times \text{SO}(\mathcal{N})} \quad (5.8)$$

Consider the Maurer-Cartan equations of the supergroup $\text{Osp}(\mathcal{N}|4)$ which can be written as follows:

$$\begin{aligned} d\omega^{ab} + \omega^{ac} \wedge \omega_c^b + 16e^2 E^a \wedge E^b &= -i 2e \bar{\Psi}_A \wedge \gamma^{ab} \gamma^5 \psi_A, \\ dE^a + \omega_c^a \wedge E^c &= i \frac{1}{2} \bar{\psi}_A \wedge \gamma^a \psi_A, \\ d\psi_A - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab} \psi_A + e \mathcal{A}_{AB} \wedge \psi_B &= 2e E^a \wedge \gamma_a \gamma_5 \psi_A, \\ d\mathcal{A}_{AB} + e \mathcal{A}_{AC} \wedge \mathcal{A}_{CB} &= 4i \bar{\psi}_A \wedge \gamma_5 \psi_B, \end{aligned} \quad (5.9)$$

where ω^{ab} , the spin-connection, is dual to the generators J_{ab} of $\text{SO}(1,3)$, E^a , the vielbein is dual to the translation generators P_a of $\text{SO}(2,3)$, \mathcal{A}_{AB} , the R-symmetry connection is dual to the generators T_{AB} of $\text{SO}(\mathcal{N})$ and finally ψ_A , the gravitino, is dual to the \mathcal{N} supersymmetry charges Q_A . Suppose that you have constructed a solution of these Maurer-Cartan equations on the coset (5.8) using some parametrization of the latter in terms of four bosonic coordinates x^a and $4 \times \mathcal{N}$ fermionic coordinates θ_A . Then the general recipe to construct the $D = 11$ vielbein V^a and the $D = 11$ gravitino Ψ is the following [20]

$$\begin{aligned} V^a &= \begin{cases} V^a &= E^a \\ V^\alpha &= \mathcal{B}^\alpha + \frac{1}{8} \sum_{A,B} \bar{\eta}_A \tau^\alpha \eta_B \mathcal{A}_{AB} \end{cases} \\ \Psi &= \sum_A \eta_A \otimes \psi_A \end{aligned} \quad (5.11)$$

The only other item which is necessary in order to write the explicit form of the supermembrane action is an explicit parametrization of the three-form $A^{[3]}$. As it was noted in [20], while the curvature $\mathbf{F}^{[4]}$ can be written intrinsically in terms of the above geometrical data for any background of the considered type, the corresponding gauge potential $\mathbf{A}^{[3]}$ can be explicitly solved only within an explicit parametrization of the supercoset (5.8). In [21] the solution of this problem was explicitly found for the case of the seven sphere by using the so named supersolvable parametrization of the coset. We refer the reader to [21] for all the details and we just quote the result. The bosonic coordinates of AdS_4 are named (ρ, t, w, x) and the AdS metric is written as follows:

$$ds^2 = \rho^2 (-dt^2 + dx^2 + dw^2) + \frac{R^2}{4} \frac{1}{\rho^2} d\rho^2 \quad (5.12)$$

The parametrizations of the vielbeins in terms of these bosonic coordinates and of the eight four-dimensional fermionic ones (θ_α^A) is the following

$$\begin{aligned} E^0 &= -\rho dt - 2e\rho\bar{\theta}^A\gamma^0 d\theta^A, \\ E^1 &= \rho dw - 2e\rho\bar{\theta}^A\gamma^1 d\theta^A, \\ E^2 &= \rho dx - 2e\rho\bar{\theta}^A\gamma^3 d\theta^A, \\ E^3 &= \frac{R}{2}\frac{1}{\rho}d\rho, \end{aligned} \tag{5.13}$$

and for the gravitino we have:

$$\psi^A = \sqrt{2e\rho} \begin{pmatrix} 0 \\ 0 \\ d\theta_1^A \\ d\theta_2^A \end{pmatrix}, \tag{5.14}$$

where $\theta^A = \frac{1-\gamma^5\gamma^2}{2}\theta^A$ and $\bar{\theta}^A = \theta^A\gamma^0$. It can also be found that the SO(8) connection \mathcal{A} , in this parametrization, is identically zero:

$$\mathcal{A}_{AB} = 0. \tag{5.15}$$

and in this parametrization the three-form can be written as follows:

$$A^{[3]} = \frac{1}{6} E^i \wedge E^j \wedge E^k \epsilon_{kji} - \frac{1}{2e} \sum_{A,B} \mathcal{B}^\alpha \wedge \eta_A \tau_\alpha \eta_B \bar{\psi}_A \wedge \psi_B. \tag{5.16}$$

where the index $i = 0, 1, 2$.

Using these data inside the general formulae presented in this article the quantum action of the supermembrane on the background $\text{AdS}_4 \times \text{S}^7$, becomes fully explicit.

6 Conclusions

We have shown that the pure spinor supermembrane has a nice geometrical structure and we have shown how to use it in order to consider the theory on any background. The geometrical structure uncovered by our analysis is deeply rooted in the structure of the Free Differential Algebra of M-theory and eventually in the cohomology structure of the $D = 11$ super Poincaré algebra from which the FDA streams.

However, this is only a starting point towards a more complete analysis. There are several problems that can be tackled using the action presented here. Just to mention some of them:

1. One loop computations of the supermembrane instanton contributions to the 11d superpotential (see [35]),
2. Compactifications on manifolds of G_2 holonomy or of weak G_2 -holonomy,

3. Amplitudes and spectrum.
4. Extension of our methods to other p -branes, in particular the $D3$ -brane and the $M5$ -brane

We hope to report on some of them soon.

A A Note on Free Differential Algebras

The algebraic structure that goes under the name of Free Differential Algebra was independently discovered at the beginning of the eighties in Mathematics by Sullivan [31] and in Physics by one of the present authors (P.F.) in collaboration with R. D'Auria [25]. Free Differential Algebras (FDA) are a categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (p -forms) of rank greater than one. The very existence of FDA.s is a consequence of Chevalley cohomology of ordinary Lie algebras and Sullivan has provided us with a very elegant classification scheme of these algebras based on two structural theorems rooted in the set up of such an elliptic complex. As it was already noted about two decades ago in [26], FDA.s have the additional fascinating property that, differently from ordinary Lie algebras they already encompass their own gauging. Indeed the first of Sullivan's structural theorems, which is in some sense analogous to Levi's theorem for Lie algebras, states that the most general FDA is a semi-direct sum of a so called minimal algebra \mathbb{M} with a contractible one \mathbb{C} . The generators of the minimal algebra are physically interpreted as the connections or *potentials*, while the contractible generators are physically interpreted as the *curvatures*. The real hard-core of the FDA is the minimal algebra and it is obtained by setting the contractible generators (the curvatures) to zero. The structure of the minimal algebra \mathbb{M} , on its turn, is beautifully determined by Chevalley cohomology of \mathbb{G} . This happens to be the content of Sullivan's second structural theorem. A recent review of FDAs also in relation with compactifications is contained in [32]. Other recent work on the topic is contained in [36] and in [37].

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